CHAPTER 15

Decision Theory and Decision Tree Analysis

MATHS DEPT.

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Chapter Outline

- Introduction
- Decision-Making Environments
- Posterior Probabilities and Bayesian Analysis
- Decision Tree Analysis
- Solved Examples
- Review Questions
- Self Practice Problems
- Hints and Answers

INTRODUCTION

The success or failure that an individual or organization experiences, depends to a large extent on the ability of making appropriate decisions. Making of a decision requires an enumeration of feasible and viable alternatives (courses of action or strategies), the projection of consequences associated with different alternatives, and a measure of effectiveness (or an objective) by which the most preferred alternative is identified. *Decision theory* provides an analytical and systematic approach to the study of decision-making. In other words, decision theory provide a method of natural decision-making wherein data concerning the occurrence of different outcomes (consequences) may be evaluated to enable the decision-maker to identify suitable alternative (or course of action).

Decision models useful in helping decision-makers make the best possible decisions are classified according to the *degree of certainty.* The scale of certainty can range from complete certainty to complete uncertainty. The region which falls between these two extreme points corresponds to the decision-making under risk (probabilistic problems).

Irrespective of the type of decision model, there are certain essential characteristics which are common to all as listed below.

Decision alternatives There is a finite number of decision alternatives available with the decision-maker at each point in time when a decision is made. The number and type of such alternatives may depend on the previous decisions made and on what has happened subsequent to those decisions. These alternatives are also called *courses of action* (*actions, acts or strategies*) and are under control and known to the decision-maker. These may be described numerically such as, stocking 100 units of a particular item, or non-numerically such as, conducting a market survey to know the likely demand of an item.

State of nature A possible future condition (consequence or event) resulting from the choice of a decision alternative depends upon certain factors beyond the control of the decision-maker. These factors are called states of nature (future). For example, if the decision is to carry an umbrella or not, the consequence (get wet or do not) depends on what action nature takes.

The states of nature are mutually exclusive and collectively exhaustive with respect to any decision problem. The states of nature may be described numerically such as, demand of 100 units of an item or non-numerically such as, employees strike, etc.

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Payoff A numerical value resulting from each possible combination of alternatives and states of nature is called payoff. The payoff values are always conditional values because of unknown states of nature. A tabular arrangement of these conditional outcome (payoff) values is known as *payoff matrix* as shown in Table 15.1.

Table 15.1 General Form of Payoff Matrix

States of Nature	*Courses of Action (Alternatives)* S_1	S_2	$\cdots$	S_n
N_1	p_{11}	p_{12}	$\cdots$	p_{1n}
N_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
N_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

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DECISION-MAKING ENVIRONMENTS

Decisions are made based upon the data available about the occurrence of events as well as the decision situation (or environment). There are four types of decision-making environment: *Certainty*, *uncertainty*, *risk* and *conflict*.

Decision-making under certainty In this case the decision-maker has the complete knowledge (perfect information) of consequence of every decision choice (course of action or alternative) with certainty. Obviously, he will select an alternative that yields the largest return (payoff) for the known future (state of nature). For example, the decision to purchase either NSC (National Saving Certificate); Indira Vikas Patra, or deposit in NSS (National Saving Scheme) is one in which it is reasonable to assume complete information about the future because there is no doubt that the Indian government will pay the interest when it is due and the principal at maturity.

Decision-making under uncertainty In this case the decision-maker is unable to specify the probabilities with which the various states of nature (futures) will occur. Thus, decisions under uncertainty are taken with even less information than decisions under risk. For example, the probability that Mr X will be the prime minister of the country 15 years from now is not known.

Several methods for arriving at an optimal solution under uncertainty are discussed below:

Criterion of Optimism (maximax or minimin) The working method is summarized as follows:

(a) Locate the maximum (or minimum) payoff values corresponding to each alternative (or course of action), then

(b) Select an alternative with best anticipated payoff value (maximum for profit and minimum for cost).

Since in this criterion the decision-maker selects an alternative with largest (or lowest) possible payoff value, it is called an *optimistic decision criterion*.

Criterion of Pessimism (minimax or maximin) The working method is summarized as follows:

(a) Locate the minimum (or maximum in case of profit) payoff in case of loss (or cost) values corresponding to each alternative, then

(b) Select an alternative with the best anticipated payoff value (maximum for profit and minimum for loss or cost).

Since in this criterion the decision-maker is conservative about the future and always anticipate worst possible outcome (minimum for profit and maximum for cost or loss), it is called a *pessimistic decision criterion*. This criterion is also known as Wald's criterion.

Equally Likely Decision (Laplace) Criterion The working method is summarized as follows:

(a) Assign equal probability value to each state of nature by using the formula:

1/(number of states of nature).

(b) Compute the expected (or average) payoff for each alternative (course of action), by adding all the payoffs and dividing by the number of possible states of nature or by applying the formula: probability of state of nature j and p_{ij} payoff value for the combination of alternative and state of nature j.

(c) Select the best expected payoff value (maximum for profit and minimum for cost).

Criterion of Realism (Hurwicz criterion) The Hurwicz approach suggest that the decision-maker must select an alternative that maximizes

$$H \text{ (criterion of realism)} = \alpha \text{ (Maximum in column)} + (1 - \alpha)\text{(Minimum in column)}$$

The working method is summarized as follows:

(a) Decide the coefficient of optimism α and then coefficient of pessimism $(1 - \alpha)$.

(b) For each alternative select the largest and lowest payoff value and multiply these with α and $(1 - \alpha)$ values, respectively. Then calculate the weighted average, H by using above formula.

(c) Select an alternative with best anticipated weighted average payoff value.

Criterion of Regret (savage criterion) The working method is summarized as follows:

(a) From the given payoff matrix, develop an opportunity-loss (or regret) matrix.
  (i) find the best payoff corresponding to each state of nature, and
  (ii) subtract all other entries (payoff values) in that row from this value.

(b) For each course of action (strategy) identify the worst or maximum regret value. Record this number as a new row.

(c) Select the course of action (alternative) with the smallest anticipated opportunity-loss value.

Decision-making under risk In this case the decision-maker has less than complete knowledge with certainty of the consequence of every decision choice (course of action). This means there is more than one state of nature (future) and for which he makes an assumption of the probability with which each state of nature will occur. For example, probability of getting head in the toss of a coin is 0.5.

Expected Monetary Value (EMV) The expected monetary value (EMV) for a given course of action is the weighted average payoff, which is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature. Mathematically EMV is stated as follows:

$$\text{EMV (Course of action, } S_j) = \sum_{i=1}^{m} p_{ij}\, p_i$$

where m = number of possible states of nature

p_i = probability of occurrence of state of nature i

p_{ij} = payoff associated with state of nature N_i and course of action, S_j

Steps for calculating EMV The various steps involved in the calculation of EMV are as follows:

(a) Construct a payoff matrix listing all possible courses of action and states of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature.

(b) Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and add these weighted values for each course of action.

(c) Select the course of action that yields the optimal EMV.

Expected Opportunity Loss (EOL) An alternative approach to maximizing expected monetary value (EMV) is to minimize the expected opportunity loss (EOL), also called *expected value of regret*. The EOL is defined as the difference between the highest profit (or payoff) for a state of nature and the actual profit obtained for the particular course of action taken. In other words, EOL is the amount of payoff that is lost by not selecting the course of action that has the greatest payoff for the state of nature that actually occur. The course of action due to which EOL is minimum is recommended.

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$$H\ (criterion\ of\ realism) = \alpha\ (Maximum\ in\ column) + (1-\alpha)(Minimum\ in\ column)$$

The working method is summarized as follows:

(a) Decide the coefficient of optimism α and then coefficient of pessimism $(1-\alpha)$.

(b) For each alternative select the largest and lowest payoff value and multiply these with α and $(1-\alpha)$ values, respectively. Then calculate the weighted average, H by using above formula.

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Expected Opportunity Loss (EOL) An alternative approach to maximizing expected monetary value (EMV) to minimize the expected opportunity loss (EOL), also called *expected value of regret*. The EOL is defined as the difference between the highest profit (or payoff) for a state of nature and the actual profit obtained for the particular course of action taken. In other words, EOL is the amount of payoff that is lost by not selecting the course of action that has the greatest payoff for the state of nature that actually occur. The course of action due to which EOL is minimum is recommended.

Since EOL is an alternative decision criterion for decision-making under risk, therefore, the results will always be the same as those obtained by EMV criterion discussed earlier. Thus, only one of the two methods should be applied to reach a decision. Mathematically, it is stated as follows:

$$\text{EOL}\,(\textit{State of nature}, N_i) = \sum_{i=1}^{m} l_{ij}\, p_i$$

where l_{ij} = opportunity loss due to state of nature, N_i and course of action, S_j

p_i = probability of occurrence of state of nature, N_i

Steps for calculating EOL Various steps involved in the calculation of EOL are as follows:

(a) Prepare a conditional profit table for each course of action and state of nature combination along with the associated probabilities.
(b) For each state of nature calculate the conditional opportunity loss (COL) values by subtracting each payoff from the maximum payoff for that outcome.
(c) Calculate EOL for each course of action by multiplying the probability of each state of nature with the COL value and then adding the values.
(d) Select a course of action for which the EOL value is minimum.

Incremental (Marginal) Analysis In many decision making problems, the calculations involved in conditional profit and expected profit become difficult. This excessive computational work can be avoided by *incremental or (marginal) approach of decision making*. According to this approach, any additional unit purchased will either be sold or remain unsold. If p represents the probability of selling one additional unit, then $(1 - p)$ must be the probability of not selling it. If the additional unit is sold, the conditional profit will increase as a result of the profit earned from this unit. This is termed as *incremental (marginal) profit*, (IP). If the additional unit is not sold, the conditional profit reduces and the amount of reduction is called the *incremental (marginal) loss*, (IL) resulting from stocking of item that is not sold.

Additional units should be stoked as long as the expected incremental profit from stocking each of them is more than the expected incremental loss. The expected incremental profit from stoking and selling an additional unit is the incremental profit of the unit multiplied by the probability that the unit would be sold, i.e. p(IP). The expected incremental loss from stoking and not selling an additional unit is the incremental loss if the unit remains unsold multiplied by the probability that the unit would not be sold, i.e. $(1 - p)$(IL). Thus the unit should be stocked upto the point where

$$p(\text{IP}) = (1-p)\,(\text{IL}) \quad \text{or} \quad p(\text{IP}+\text{IL}) = \text{IL} \quad \text{or} \quad p = \frac{\text{IL}}{\text{IP}+\text{IL}}$$

where p is the required probability of selling at least an additional unit of an item to justify the stoking of that additional unit. Additional units should be stoked as long as the probability of selling at least an additional unit, is greater than p.

POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

An initial probability statement used to evaluate expected payoff is called a *prior* probability distribution. The one which has been revised in the light of information which has come to hand is called a *posterior* probability distribution. It will be evident that what is a posterior to one sequence of state of nature becomes the prior to others which are yet to happen.

Let $A_1, A_2, \ldots, A_n$ are mutually exclusive and collectively exhaustive outcomes. Their prior probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ are given. There is an experimental outcome B for which the conditional probabilities $P(B \mid A_1), P(B \mid A_2), \ldots, P(B \mid A_n)$ are also known. Given the information that outcome B has occurred, then the revised probabilities $P(A_i \mid B)$, $i = 1, 2, \ldots, n$ are determined by using following conditional probability relationship:

$$P(A_i \cap B) = P(A_i) \times P(B \mid A_i)$$

$$P(A_i \mid B) = \frac{P(A_i)\,P(B \mid A_i)}{P(A_1)\,P(B \mid A_1) + P(A_2)\,P(B \mid A_2) + \ldots + P(A_n)\,P(B \mid A_n)}$$

DECISION TREE ANALYSIS

The decision tree analysis involves construction of a diagram showing all the possible courses of action, states of nature and the probabilities associated with the states of nature. The '*decision diagram*' looks very much like a drawing of a tree, therefore, also called '*decision-tree*'.

A decision tree consists of *nodes, branches, probability estimates*, and *payoffs*. There are two types of nodes: *decision nodes* and *chance nodes*. A decision node is usually represented by a *square* □ and indicates places where a decision-maker must make a decision. Each branch leading away from a decision node represents one of the several possible courses of action available to the decision-maker. The chance node is represented by a *circle* ○ and indicates a point at which the decision-maker will discover the response to his decision, i.e. different possible outcomes (states of nature, competitors actions, etc.) which can result from a chosen course of action.

Branches emanate from and connect various nodes are either decisions or states of nature. There are two types of branches: *decision branches* and *chance branches*. Each branch leading away from a decision node represents a course of action or strategy that can be chosen at this decision point, whereas a branch leading away from a chance node represents the state of nature of a set of chance factors. Associated probabilities are indicated along side of respective chance branch. These probabilities are the likelihood that the chance outcome will assume the value assigned to the particular branch. Any branch that makes the end of the decision tree, i.e. it is not followed by either a decision or chance node, is called a *terminal branch*. A terminal branch can represent either a course of action or a chance outcome. The terminal points of a decision tree are said to be mutually exclusive points so that exactly one course of action will be chosen.

SOLVED EXAMPLES

Example 15.1 A firm manufactures three types of products. The fixed and variable costs are given below:

		Fixed Cost (Rs.)	*Variable Cost per Unit (Rs.)*
Product *A*	:	25,000	12
Product *B*	:	35,000	9
Product *C*	:	53,000	7

The likely demand (units) of the products is given below:

Poor demand	:	3,000
Moderate demand	:	7,000
High demand	:	11,000

If the sale price of each type of product is Rs. 25, then prepare the payoff matrix.

Solution Let D_1, D_2 and D_3 be the poor, moderate and high demand, respectively. Then payoff is given by

Payoff = Sales revenue – Cost

The calculations for payoff (in thousand, Rs.) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$D_1 A = 3 \times 25 - 25 - 3 \times 12 = 14$ $\quad$ $D_2 A = 7 \times 25 - 25 - 7 \times 12 = 66$

$D_1 B = 3 \times 25 - 35 - 3 \times 9 = 13$ $\quad$ $D_2 B = 7 \times 25 - 35 - 7 \times 9 = 77$

$D_1 C = 3 \times 25 - 53 - 3 \times 7 = 1$ $\quad$ $D_2 C = 7 \times 25 - 53 - 7 \times 7 = 73$

$D_3 A = 11 \times 25 - 25 - 11 \times 12 = 118$

$D_3 B = 11 \times 25 - 35 - 11 \times 9 = 141$

$D_3 C = 11 \times 25 - 53 - 11 \times 7 = 145$

The payoff values are shown in Table 15.2.

Table 15.2 ('000 Rs.)

Product Type	*Alternative Demand* D_1	D_2	D_3
A	14	66	
B	13	77	
C	1	73	

Example 15.2 A food products company is contemplating the introduction of a new with new packaging or replace the existing product at much higher price (S_1) or a composition of the existing product with a new packaging at a small increase in price the composition of the existing product except the word 'New' with a negligible increase possible states of nature or events are: (i) high increase in sales (N_1), (ii) no change decrease in sales (N_3). The marketing department of the company worked out the payoff profits for each of the strategies of three events (expected sales). This is represented

Strategies	*States of Nature* N_1	N_2	N_3
S_1	7,00,000	3,00,000	1,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000	3,00,000	3,00,000

Which strategy should the concerned executive choose on the basis of

(i) Maximin criterion (ii) Maximax criterion

(iii) Minimax regret criterion (iv) Laplace criterion?

Solution The payoff matrix is rewritten as follows:

(i) *Maximin Criterion:*

States of Nature	*Strategies* S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column minimum	1,50,000	0	**3,00,000**

The maximum of column minima is 3,00,000. Hence, the company should adopt

(ii) *Maximax Criterion:*

States of Nature	*Strategies* S_1	S_2	S_3
N_1	7,00,000	5,00,000	3,00,000
N_2	3,00,000	4,50,000	3,00,000
N_3	1,50,000	0	3,00,000
Column maximum	***7,00,000*** ↑ *Maximax*	5,00,000	3,00,000

The maximum of column maxima is 7,00,000. Hence, the company should adopt

(iii) *Minimax Regret Criterion:* Opportunity loss table is shown below:

States of Nature	*Strategies* S_1	S_2	S_3
N_1	7,00,000 – 7,00,000 = 0	7,00,000 – 5,00,000 = 2,00,000	
N_2	4,50,000 – 3,00,000 = 1,50,000	4,50,000 – 4,50,000 = 0	
N_3	3,00,000 – 1,50,000 = 1,50,000	3,00,000 – 0 = 3,00,000	
Column maximum	***1,50,000*** ↑ *Minimax regret*	3,00,000	